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Project RAND

ON THE HAMILTONIAN GAME
(A Traveling Salesmen Problem)

Julia Robinson

RM-303

5 December 1949

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The RAND Corporation

SANTA MONICA · CALIFORNIA

PROJECT RAND

RESEARCH MEMORANDUM

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1. Introduction

The purpose of this note is to give a method for solving a problem related to the traveling salesman problem. It seems worthwhile to give a description of the original problem.* One formulation is to find the shortest route for a salesman starting from Washington, visiting all the state capitals and then returning to Washington.** More generally, to find the shortest closed curve containing n given points in the plane.

Clearly, it is sufficient to consider curves made up of line segments joining pairs of the given points. Also, unless all the points lie on a straight line, the optimal path will not pass through any point twice. Hence the problem can be stated as follows:

> Arrange the n points in a cyclic order so that the sum of the distance between consecutive points is a minimum.

In this statement of the problem, arbitrary real numbers can be assigned as the "distances" between ordered pairs of distinct points. Thus, the "distance" from A to B need not be the same as from B to A. We shall sometimes refer to the "length" of AB instead of the "distance" from A to B.

Since there are only a finite number of paths to consider, the problem consists in finding a method for picking out the optimal path when n is moderately large, say n = 50. In this case, there are more than 10^{62} possible paths, so we can not simply try them all. Even for as few as 10 points, some short cuts are desirable.

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* Actually, the problem may go back to W. R. Hamilton. See R. W. Ball: <u>Mathematical Recreations</u> on the Hamiltonian game.

** In this paper I shall not be concerned with the various possible applications of the problem solved here.

2. Statement of the problem

An unsuccessful attempt to solve the above problem led to a solution of the following:

Given n points and all the "distances" between ordered pairs of distinct points. The problem is to find a system of ordered circuits such that:

i. Each point lies on exactly one circuit.

ii. Each circuit contains at least 2 points.

iii. No circuit passes through the same point more than once.

iv. The total "length" of the circuits is a minimum.

However at first glance, it looks more difficult than the traveling salesman problm, for there are obviously many more systems of circuits than circuits. Actually the topological characterization of a system of circuits is much simpler than that of a single circuit and can be used to solve this problem.

The method presented here of handling this problem will enable us to check whether a given system of circuits is optimal or, if not, to find a better one. I believe it would be feasible to apply it to as many as 50 points provided suitable calculating equipment is available.

3. Description of the method.

Number the points 1, 2, ..., n. Put $D = \|d_{ij}\|$, where d_{ij} is the distance from i to j, $d_{ii} = 0$. Let $\mathscr{S}$ be the set of directed segments comprising the proposed system of circuits. We wish to determine if this system is optimal or, if not, to find a better system.

Construct the auxiliary matrix $S = \|s_{ij}\|$ as follows:

For each i, determine i' so that $ii' \in \mathscr{S}$. Then put,

$$s_{ii'} = +\infty$$

and

$$s_{ij} = d_{ji'} - d_{ii'} \qquad \text{for } j \neq i' .$$

Now think of the S-matrix as giving new "distances" between the given points and look for a closed circuit of negative S-length. If there is such a circuit, it will have from 2 to n points. Suppose $\mathcal{C} = i_0 i_1 \ldots i_k$ is a circuit of negative S-length. Then make up a new system of circuits $\mathcal{S}'$ by modifying $\mathcal{S}$ in the following way:

Remove	Add
$i_o i_o'$	$i_1 i_o'$
$i_1 i_1'$	$i_2 i_1'$
.	.
.	.
.	.
$i_k i_k'$	$i_o i_k'$

The new system of circuits $\mathcal{S}'$, thus obtained, has a shorter total D-length than $\mathcal{S}$. In fact, if we let $\ell_A(\mathcal{Q})$ be the length of $\mathcal{Q}$ measured by the matrix A, then

$$\ell_D(\mathcal{S}') = \ell_D(\mathcal{S}) + \ell_S(\mathcal{C}) .$$

We then apply the same procedure to $\mathcal{S}'$.

Suppose we can not find a circuit of negative S-length. Then we attempt to show that $\mathcal{S}$ is optimal. To do this, enforce the triangle inequality,

$s_{ij} \leq s_{ik} + s_{kj}$;* that is, if $s_{ij} > s_{ik} + s_{kj}$ replace s_{ij} by $s_{ik} + s_{kj}$. These replacements can be carried out in any order.

If a matrix is eventually obtained for which the triangle inequality holds, then $\mathcal{S}$ is the best system of circuits. If not, there must be some circuit of negative S-length. To find one, we must keep track of the changes made in the S-matrix. For example, under the i, j^th entry in the S-matrix, write (ij). Then if s_{ij} is replaced by $s_{ik} + s_{kj}$, replace the (ij) by (ikj). Similarly, if s_{iKj} is replaced by $s_{iH\ell} + s_{\ell Mj}$, then replace (iKj) by (iHℓMj). (Here K, H and M are finite sequences of numbers from 1 to n.) Thus, the entry in the i, j^th place will always be the length of the path indicated from i to j. If there is a negative circuit in the S-matrix, then at some stage a negative number can be put on the main diagonal of the modified S-matrix. We can then easily obtain the corresponding circuit in the S-matrix.

4. A numerical example

As an example, we take a set of six points with the following distance matrix:

$$\begin{matrix} 0 & 1 & 4 & 2 & 8 & 7 \\ 6 & 0 & 5 & 2 & 1 & 9 \\ 4 & 8 & 0 & 7 & 2 & 6 \\ 5 & 5 & 5 & 0 & 4 & 8 \\ 6 & 1 & 5 & 7 & 0 & 4 \\ 3 & 9 & 1 & 2 & 6 & 0 \end{matrix}$$

D

* i, j and k need not be distinct.

As a first trial system of circuits $\mathcal{S}$ take the two circuits 12531 and 464. Then $\mathcal{S} = \{12, 25, 53, 31, 46, 64\}$. Hence 1' = 2, 2' = 5, 3' = 1, 4' = 6, 5' = 3 and 3' = 1. Next construct the S-matrix:

0	+∞	+7	+4	0	+8
+7	0	+1	+3	+∞	+5
+∞	+2	0	+1	+2	-1
-1	+1	-2	0	-4	+∞
-1	0	+∞	0	0	-4
0	0	+5	+∞	+5	0

S

We now look for a closed circuit of negative S-length. After a few trials, we find the circuit $\mathcal{C}$ = 456234 with S-length = -6. We obtain $\mathcal{S}'$ from $\mathcal{S}$ by removing 46, 53, 64, 25 and 31 from $\mathcal{S}$ and adjoining 56, 63, 24, 35 and 41. We then obtain the $\mathcal{S}'$ matrix:

0	+∞	+7	+4	0	+8
0	0	+5	+∞	+5	0
+6	-1	0	+2	+∞	+4
+∞	+1	-1	0	+1	-2
+3	+5	+2	+4	0	+∞
+3	+4	+∞	+4	+4	0

S'

Since we do not find a negative circuit in S', we try to enforce the triangle inequality, keeping track of the changes we make in case there is a negative circuit. We give one intermediate matrix as an example and the final one in which the triangle inequality holds.

0 (11)	+5 (142)	+2 (153)	+4 (14)	0 (15)	+2 (146)
0 (21)	0 (22)	+2 (2153)	+4 (214)	0 (215)	0 (26)
-1 (321)	-1 (32)	0 (33)	+2 (34)	-1 (3215)	-1 (326)
-2 (4321)	-2 (432)	-1 (43)	0 (44)	-2 (43215)	-2 (46)
+1 (5321)	+1 (532)	+2 (53)	+4 (54)	0 (55)	+1 (5326)
+2 (64321)	+2 (6432)	+3 (643)	+4 (64)	+2 (643215)	0 (66)

Intermediate modified matrix

0	+1	+2	+4	0	+1
(11)	(1532)	(153)	(14)	(15)	(15326)
0	0	+2	+4	0	0
(21)	(22)	(2153)	(214)	(215)	(26)
-1	-1	0	+2	-1	1
(321)	(32)	(33)	(34)	(3215)	(326)
-2	-2	-1	0	-2	-2
(4321)	(432)	(43)	(44)	(43215)	(46)
+1	+1	+2	+4	0	+1
(5321)	(532)	(53)	(54)	(55)	(5326)
+2	+2	+3	+4	+2	0
(64321)	(6432)	(643)	(64)	(643215)	(66)

Final S'-matrix with $\triangle$ -inequality holding

Hence $\mathcal{S}'$ is the optimal system of circuits. It consists of the two circuits 1241 and 3563 and has D-length = 15.

5. Justification of the method.

First, notice that a set of n directed segments satisfies i - iii of Section 1, if and only if

1. Each of the n points is an initial point of one of the segments;
2. Each of the n points is a terminal point of one of the segments;
3. Each segment is between distinct points.

To see this, think of the terminal points as a permutation of the initial points. This permutation can be expressed as a product of cyclic permutations. These are the circuits.

This insures that, if there is a circuit $\mathcal{C}$ of negative S-length and if $\mathcal{S}'$ is obtained from $\mathcal{S}$ by the rule given in Section 3, then $\mathcal{S}'$ will also be an admissible system of circuits. This is clear since, if a segment with initial point a is removed, one is also added and conversely. Similarly, for the terminal points. Hence 1 and 2 remain satisfied. Furthermore, if $\mathcal{C}$ is of negative S-length, it can not contain any segments in common with $\mathcal{S}$, for these have S-length $+\infty$; therefore, the segments added to $\mathcal{S}$ are between distinct points.

Let $\mathcal{C} = i_0 i_1 \ldots i_k$. Then

$$\ell_S(\mathcal{C}) = s_{i_0 i_1} + s_{i_1 i_2} + \ldots + s_{i_k i_0}$$

$$= (d_{i_1 i_0'} - d_{i_0 i_0'}) + (d_{i_2 i_1'} - d_{i_1 i_1'}) + \ldots + (d_{i_0 i_k'} - d_{i_k i_k'})$$

$$= (d_{i_1 i_0'} + d_{i_2 i_1'} + \ldots + d_{i_0 i_k'}) - (d_{i_0 i_0'} + d_{i_1 i_1'} + \ldots + d_{i_k i_k'}).$$

Hence $\ell_D(\mathcal{S}') - \ell_D(\mathcal{S}) = \ell_S(\mathcal{C})$. Thus, we see that if there is a circuit $\mathcal{C}$ of negative S-length, then $\mathcal{S}$ is not an optimal system and we can construct a system $\mathcal{S}'$ of shorter total length.

Conversely, if $\mathcal{S}$ is not optimal, then we will show that there is a circuit

$\mathcal{C}$ of negative S-length. Let $\mathcal{S}'$ be a system of circuits of shorter total D-length than $\mathcal{S}$. Let $\mathcal{a}$ be the set of segments in $\mathcal{S}$ but not in $\mathcal{S}'$ and $\mathcal{b}$ be the set of segments in $\mathcal{S}'$ but not in $\mathcal{S}$. Let $\mathcal{a} = \{i_o i_o', i_1 i_1', \ldots, i_k i_k'\}$. Then $\mathcal{b}$ must consist of a set of segments with the same initial points as in $\mathcal{a}$, with the same terminal points and the same number of segments. Hence let $\mathcal{b} = \{ j_o i_o', \ldots, j_k i_k' \}$, where $j_o, j_1, \ldots, j_k$ is a permutation of $i_o, i_1, \ldots, i_k$. Then

$$\ell_D(\mathcal{S}') - \ell_D(\mathcal{S}) = (d_{j_o i_o'} - d_{i_o i_o'}) + (d_{j_1 i_1'} - d_{i_1 i_1'}) + (d_{j_k i_k} - d_{i_k i_k'})$$

$$= s_{i_o j_o} + s_{i_1 j_1} + \cdots + s_{i_k j_k} \ .$$

Express the permutation $\begin{pmatrix} i_o i_1 \ldots i_k \\ j_o j_1 \ldots j_k \end{pmatrix}$ as the product of cycles, say $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t$. Then by rearranging and collecting terms of

$$s_{i_o j_o} + s_{i_1 j_1} + \cdots + s_{i_k j_k} \ ,$$

we see that this sum is just

$$\ell_S(\mathcal{C}_1) + \cdots + \ell_S(\mathcal{C}_t)$$

where $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t$ are the circuits corresponding to the cycles of the permutation. Hence

$$\ell_D(\mathcal{S}') - \ell_D(\mathcal{S}) = \ell_S(\mathcal{C}_1) + \cdots + \ell_S(\mathcal{C}_t).$$

Since $\mathcal{S}'$ is shorter than $\mathcal{S}$, one of the circuits $\mathcal{C}_1, \ldots, \mathcal{C}_t$ must have negative S-length. Therefore $\mathcal{S}$ is optimal if and only if there is no closed circuit of negative S-length.

It remains to show that the non-existence of a circuit of negative S-length is equivalent to the existence of a modified S-matrix for which the triangle inequality holds. Assume first that A is a modified S-matrix and that the triangle inequality holds in A. Let $\mathcal{C}$ be a circuit of negative S-length. It corresponds to a circuit $\mathcal{C}'$ of negative A-length. Let $\mathcal{C}' = i_0 i_1 i_2 \ldots i_k$. Then $\mathcal{C}'' = i_0 i_2 \ldots i_k$ is also of negative A-length since $a_{i_0 i_2} \leq a_{i_0 i_1} + a_{i_1 i_2}$. Hence, if there is any circuit of negative A-length we can find a one-point circuit of negative length i.e. for some i, $a_{ii} < 0$. But this is impossible since then $a_{ii} + a_{ii} < a_{ii}$ contrary to the assumption that the triangle inequality holds.

On the other hand, if there is no circuit of negative S-length we can enforce the triangle inequality. The resulting matrix will give in the i, j^{th} place the S-length of the shortest path from i to j. If there is no circuit of negative length, there clearly is a shortest path between any two points.

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